There are two Parts, assigned 2 point each, for a total of 4 points.

Part I (2 points):
A feedback system is shown in Figure 1 where $P(s)$ is a system model and $e^{-\tau s}$ represents transmission delay in the feedback path. The system is open loop stable and its transfer func-

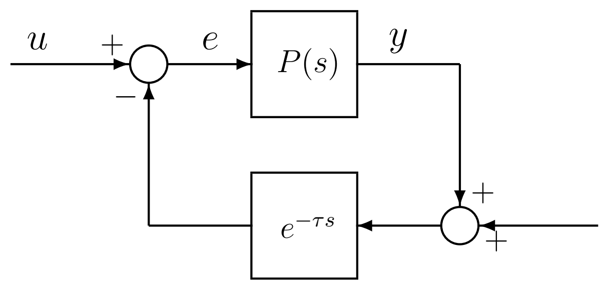


Figure 1: Feedback system.

tion can be fairly accurately modeled over the range of frequencies that are relevant to stability analysis by

$$G(s) = \sqrt{\frac{\sqrt{2}}{s+1}}$$

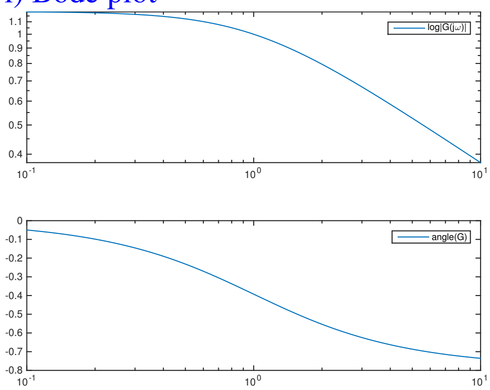
for $s = j\omega$ with ω measured in radians/sec. It is noted that the system model does have a rational transfer function, and hence it does not have a finite-dimensional state representation.

Complete/answer the following:
i) Draw the Bode plot for the open loop transfer function (approximately).
ii) Draw the Nyquist plot.
iii) Explain why the closed loop system is stable for zero time-delay $\tau = 0$.
iv) What is the maximal interval $[0, \tau_{\max})$ for the time-delay in the feedback loop for which the closed loop system remains stable.

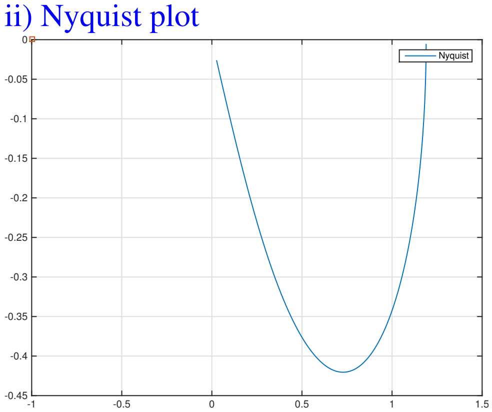
Space for your work:

Solution:

i) Bode plot



ii) Nyquist plot



iii) It is closed-loop stable because
the -1 point of the complex plane is not encircled, and
the open loop system is stable.

Using Nyquist theory, from these two facts, the number of closed loop unstable poles are then $N = Z - P = 0 - 0 = 0$.

iv) The frequency where $|G(j\omega)| = 1$ is clearly, $\omega = 1$ [rad/sec].
At that frequency, the phase is

$$\angle G(j) = -\frac{1}{2}\angle(1 + j) = -\frac{\pi}{8}.$$

Thus, the closed loop system tolerates a delay of up to $\tau_{\max} = \frac{\pi}{8}$ [sec] (since the gain crossover frequency is 1 [rac/sec]).

Part II (2 points):
Consider two $n \times n$ real matrices A and Δ.

i) Consider the autonomous linear dynamical system

$$\dot{x}(t) \;=\; (A\,+\,\Delta)\,x(t),$$

with initial conditions $x(0) = x_0$. Show that provided $A\Delta = \Delta A$, the solution is given by

$$e^{\Delta t}e^{At}x_0.$$

ii) Give an example of two matrices A, Δ such that $A\Delta \neq \Delta A$.

iii) Let $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ and $\Delta = \begin{bmatrix} 0 & \delta_2 \\ \delta_1 & 0 \end{bmatrix}$.

- Determine the conditions that the positive scalars δ_1 and δ_2 have to satisfy to guarantee stability of the above system.
- Set $\delta_2 = 0$ and find the solution to the above system for the initial condition $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
- Set $\delta_2 = 0$ and consider the system with the input u and the output y

$$\begin{aligned} \dot{x}(t) &= (A\,+\,\Delta)\,x(t) \;+\; B\,u(t) \\ y(t) &= C\,x(t) \end{aligned}$$

  where $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $C = [\ 0 \quad 1\ ]$.

  How does the peak value on the Bode magnitude plot of the transfer function (from u to y) depend on δ_1? At what frequency does this peak value take place?

Space for your work:

Solution:

i) Taking the derivative of the supposed solution we obtain:

$$\frac{d}{dt}e^{\Delta t}e^{At}x_0 = \Delta e^{\Delta t}e^{At}x_0 + e^{\Delta t}Ae^{At}x_0.$$

But since $A\Delta = \Delta A$, A commutes with powers of Δ as well

$$\Delta^2 A = \Delta A \Delta = A\Delta^2,$$

and in general, functions of Δ, i.e., $e^{\Delta t}A = Ae^{\Delta t}$. Hence,

$$\frac{d}{dt}\underbrace{e^{\Delta t}e^{At}x_0}_{z(t)} = \Delta e^{\Delta t}e^{At}x_0 + Ae^{\Delta t}e^{At}x_0$$

$$(A+\Delta)\underbrace{e^{\Delta t}e^{At}x_0}_{z(t)}.$$

This function $z(t)$ also satisfies $z(0) = x_0$, hence it is the solution to $\dot{x}(t) = (A+\Delta)x(t)$ with the given initial conditions.

ii) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\Delta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Here, $A\Delta = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ whereas $\Delta A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

iii)

- $\det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -\lambda_1 & \delta_2 \\ \delta_1 & -\lambda_2 \end{bmatrix}\right) = s^2 + (\lambda_1 + \lambda_2)\, s + \underbrace{(\lambda_1\lambda_2 - \delta_1\,\delta_2)}_{>0}$

  In our case, $\lambda_1 = 1$ and $\lambda_2 = 2$. The system is stable if $\delta_1\,\delta_2 < \lambda_1\,\lambda_2 = 2$.

- For $\delta_2 = 0$, the matrix exponential of the matrix $A + \Delta$ is given by

$$e^{(A+\Delta)t} = \begin{bmatrix} e^{-t} & 0 \\ \delta_1\,(e^{-t} - e^{-2t}) & e^{-2t} \end{bmatrix}.$$

  Thus, the desired solution is $x(t) = \begin{bmatrix} e^{-t} \\ \delta_1\,(e^{-t} - e^{-2t}) \end{bmatrix}$.

- For $\delta_2 = 0$, the transfer function from u to y is $H(s) = \dfrac{\delta_1}{(s+1)(s+2)}$. The peak value on the Bode magnitude plot takes place at $\omega = 0$ and is determined by $|H(0)| = |\delta_1|/2$.